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Efficient algorithms for construction of recurrence relations for the expansion and connection coefficients in series of Al-Salam–Carlitz I polynomials

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Abstract
Two formulae expressing explicitly the derivatives and moments of Al-Salam–Carlitz I polynomials of any degree and for any order in terms of Al-Salam–Carlitz I themselves are proved. Two other formulae for the expansion coefficients of general-order derivatives $D_q^p f(x)$, and for the moments $x^\ell D_q^p f(x)$, of an arbitrary function $f(x)$ in terms of its original expansion coefficients are also obtained. Application of these formulae for solving q-difference equations with varying coefficients, by reducing them to recurrence relations in the expansion coefficients of the solution, is explained. An algebraic symbolic approach (using Mathematica) in order to build and solve recursively for the connection coefficients between Al-Salam–Carlitz I polynomials and any system of basic hypergeometric orthogonal polynomials, belonging to the q-Hahn class, is described.

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1. Introduction

The so-called q-polynomials constitute a very important and interesting set of special functions and especially of orthogonal polynomials. In 1884, Markov introduced a family of q-polynomials. Later, in 1949, Hahn introduced the q-classical orthogonal polynomials, the q-derivatives of which are also orthogonal. This is analogous to the case of classical orthogonal polynomials, the derivatives of which are also orthogonal. In 1985, Andrews and Askey continued the work of Hahn. A collection of the formulae of the hypergeometric orthogonal polynomials which appear in the so-called Askey-scheme, as well as their q-analogues, can be found in a paper of Koekoek and Swarttouw (1998). Lately, there has been an increasing interest in the q-orthogonal polynomials. This is due to their numerous applications in several areas of mathematics, e.g., continued fractions, Eulerian series, theta functions, elliptic functions, etc (see Andrews (1986) and Fine (1988)), quantum groups and algebras (see Koornwinder (1990, 1994) and Vilenkin and Klimyk (1992)), discrete mathematics (combinatorics, graph theory), coding theory, among others (see also Gasper and Rahman (1990)). There is also a connection between the representation theory of quantum algebras (Clebsch–Gordan coefficients, $3j$ and $6j$ symbols), which play an important role in physical applications, and the q-orthogonal polynomials (see Álvarez-Nodarse *et al* (1997) and references there in).

The expansion of a given function as a series in classical orthogonal polynomials is a matter of great interest in applied mathematics and mathematical physics. This is particularly true for the connection problem between any two families of classical orthogonal polynomials. The study of such a problem has attracted a lot of interest in the last few years. Special emphasis was given to the classical continuous orthogonal polynomials (Hermite, Laguerre, Jacobi and Bessel) and the discrete cases (Charlier, Meixner, Kravchuk and Hahn). Ronveaux *et al* (1995), Godoy *et al* (1997) and Area *et al* (1998) have developed a recurrent method, called *Na ViMa* algorithm, for solving the connection problem for all families of classical orthogonal polynomials, as well as some special kind of linearization problem and used it for solving different problems related with the associated Sobolev-type polynomials, etc (see Godoy *et al* (1998a, 1998b)). Let us point out that there are very similar algorithms for finding the recurrence relations for both connection and linearization coefficients due to Lewanowicz (1996a, 1996b, 1997, 2002). Also, different algorithms for solving the connection problem for the four families of classical orthogonal polynomials of continuous variable (Laguerre, Hermite, Jacobi and Bessel) are presented by Doha (2003, 2004a, 2004b) and Doha and Ahmed (2004), respectively, and for the discrete cases (Charlier, Meixner, Kravchuk and Hahn) by Doha and Ahmed (in press and submitted).

Also, the construction of recurrence relations for the coefficients of the Fourier series expansions with respect to the q-classical orthogonal polynomials are presented by Lewanowicz (2003a, 2003b), Lewanowicz *et al* (2000) and Lewanowicz and Woźny (2001). A great importance of the connection and linearization coefficients has appeared in Álvarez-Nodarse and Ronveaux (1996), Álvarez-Nodarse *et al* (1999, 2001), Area *et al* (1999), Gasper and Rahman (1990), Lewanowicz (1998, 2000), Szwarc (1996), Ismail and Simeonov (2001).

Up to now, and to the best of our knowledge, explicit formulae for the expansion coefficients of a general-order q-derivatives of an arbitrary function and for the evaluation of the expansion coefficients of the moments of high-order q-derivatives of such function in terms of q-orthogonal polynomials—similar to those obtained by Karageorghis (1988a, 1988b), Phillips (1988), Doha (1991, 2002, 2003, 2004a, 2004b) and Doha and Ahmed (2004, in press and submitted) for classical orthogonal polynomials of continuous and discrete variables—are not known and traceless in the literature. This is in particular true for the basic hypergeometric orthogonal polynomials, belonging to the q-Hahn class, and partially motivates our interest in such polynomials. Another motivation is that the theoretical and numerical analysis of numerous physical and mathematical problems very often require the expansion of an arbitrary polynomial or the expansion of an arbitrary function with its q-derivatives and moments into a set of q-classical orthogonal polynomials. This is also true for the q-Hahn class. They are important in certain problems of mathematical physics; for example, the development in quantum groups has led to the so-called q-harmonic oscillators (see for instance, Macfarlane (1989), Biedenharn (1989), Kulish and Damaskinsky (1990) and Askey and Suslov (1993a)). The known models of q-oscillators are closely related with q-orthogonal polynomials. The q-analogues of boson operators have been introduced explicitly in Askey and Suslov (1993a), where the corresponding wave functions were constructed in terms of the continuous q-Hermite polynomials of Rogers (see Rogers (1894) and Askey and Ismail (1983, p 55)), in terms of the Stieltjes–Wigert polynomials (Stieltjes (1894), (1895) and Wigert (1923)) and in terms of q-Charlier polynomials of Al-Salam and Carlitz (1965). Askey and Suslov (1993b) have shown that Al-Salam–Carlitz I polynomials are closely connected with the q-harmonic oscillator.

The main aim of the present paper is to show that the ideas given in Doha (1991, 2002, 2003, 2004a, 2004b), Doha and Ahmed (2004, in press and submitted) can be extended to the q-orthogonal polynomials. This approach only requires the knowledge of the so-called structure and three-term recurrence relations for the q-orthogonal polynomials.

The paper is organized as follows. In section 2, we give some relevant properties of Al-Salam–Carlitz I polynomials. In section 3, we prove a theorem which relates Al-Salam–Carlitz I expansion coefficients of the q-derivatives of a function in terms of its original expansion coefficients. Explicit expressions for the q-derivatives of Al-Salam–Carlitz I polynomials of any degree and for any order as a linear combination of suitable Al-Salam–Carlitz I polynomials themselves are also deduced. In section 4, we prove a theorem which gives the Al-Salam–Carlitz I expansion coefficients of the moments of one single Al-Salam–Carlitz I polynomial of any degree. Another theorem which expresses the Al-Salam–Carlitz I expansion coefficients of the moments of a general-order q-derivative of an arbitrary function in terms of its Al-Salam–Carlitz I original expansion coefficients is also stated. In section 5, we give an application of these theorems which provides an algebraic symbolic approach (using Mathematica) in order to build and solve recursively for the connection coefficients between Al-Salam–Carlitz I polynomials and any system of basic hypergeometric orthogonal polynomials, belonging to the q-Hahn class.

2. Some properties of Al-Salam–Carlitz I polynomials

The families of q-orthogonal polynomials belonging to the q-Hahn class satisfy second-order q-difference equation, and also have the property that their derivatives form orthogonal systems. The Al-Salam–Carlitz I polynomials, $\left\{U_n^{(\alpha)}(x;q)\right\}$, a family of q-orthogonal polynomials with these two properties, were introduced by Al-Salam and Carlitz (1965). The interested reader is referred to the book of Gasper and Rahman (1990, pp 3–6) and Koekoek and Swarttouw (1998, pp 113–114), for a brief background, definitions for some terminology and most of the basic properties of Al-Salam–Carlitz I polynomials.

The following two recurrence relations (which may be found in Koekoek and Swarttouw (1998), p 113, equations (3.24.4) and (3.24.7)) are of fundamental importance in developing the present work. These are as follows:

(i) Recurrence relation

$$\begin{aligned} &xU_n^{(\alpha)}(x;q) = U_{n+1}^{(\alpha)}(x;q) + \beta_n U_n^{(\alpha)}(x;q) + \gamma_n U_{n-1}^{(\alpha)}(x;q), \qquad n \geqslant 0, \\ &U_0^{(\alpha)}(x;q) = 1, \qquad U_{-1}^{(\alpha)}(x;q) = 0, \end{aligned} \tag{1}$$

where $\beta_n = (\alpha+1)q^n$ and $\gamma_n = -\alpha q^{n-1}(1-q^n)$.

(ii) Structure formula

$$U_n^{(\alpha)}(x;q) = \frac{1}{[n+1]_q} D_q U_{n+1}^{(\alpha)}(x;q), \qquad n \geqslant 0, \tag{2}$$

where the q-derivative operator D_q and the q-analogues of the real numbers, $[x]_q$, are defined (Hahn 1949) by

$$D_q f(x) := \begin{cases} \dfrac{f(qx) - f(x)}{(q-1)x}, & x \neq 0, \\ f'(0), & x = 0, \ \text{provided } f'(0) \text{ exists}, \end{cases} \tag{3}$$

$$[x]_q := \begin{cases} \dfrac{1-q^x}{1-q}, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{4}$$

3. Relation between the coefficients $a_n^{(p)}$ and a_n and the pth q-derivative of $U_n^{(\alpha)}(x; q)$

The main result of this section is to prove the following theorem which expresses explicitly the Al-Salam–Carlitz I expansion coefficients, $a_n^{(p)}$, of a general-order q-derivative of an infinitely q-differentiable function in terms of its original Al-Salam–Carlitz I coefficients, a_n.

Theorem 1. *If we are given a regular function $f(x)$ which is formally expanded in an infinite series of Al-Salam–Carlitz I polynomials,*

$$f(x) = \sum_{n=0}^{\infty} a_n U_n^{(\alpha)}(x; q), \tag{5}$$

and for the pth *q-derivative of $f(x)$,*

$$D_q^p f(x) = \sum_{n=0}^{\infty} a_n^{(p)} U_n^{(\alpha)}(x; q), \qquad a_n^{(0)} = a_n, \tag{6}$$

then

$$a_n^{(p)} = [p]_q! \begin{bmatrix} n+p \\ n \end{bmatrix}_q a_{n+p}, \qquad n \geqslant 0, \tag{7}$$

and clearly

$$D_q^p U_n^{(\alpha)}(x; q) = [p]_q! \begin{bmatrix} n \\ n-p \end{bmatrix}_q U_{n-p}^{(\alpha)}(x; q), \qquad n, p \geqslant 0, \tag{8}$$

where

$$[p]_q! := \prod_{j=1}^{p} [j]_q = \frac{(q; q)_p}{(1-q)^p}.$$

Proof. In view of (6), we have

$$D_q^{p+1} f(x) = \sum_{n=0}^{\infty} a_n^{(p+1)} U_n^{(\alpha)}(x; q), \tag{9}$$

and on differentiating (6), and making use of (2), we get

$$D_q^{p+1} f(x) = \sum_{n=0}^{\infty} a_n^{(p)} D_q U_n^{(\alpha)}(x; q) = \sum_{n=0}^{\infty} [n+1]_q a_{n+1}^{(p)} U_n^{(\alpha)}(x; q). \tag{10}$$

From (9) and (10), we get immediately

$$a_n^{(p+1)} = [n+1]_q a_{n+1}^{(p)}, \qquad n \geqslant 0.$$

It can easily be shown that

$$a_n^{(p)} = \left[\prod_{j=1}^{p}[n+j]_q\right] a_{n+p} = [p]_q! \begin{bmatrix} n+p \\ n \end{bmatrix}_q a_{n+p}, \qquad n \geqslant 0,$$

which proves (7). The proof of formula (8) is clear, which completes the proof of theorem 1. □

4. Explicit formula for the expansion coefficients of the moments of $D_q^p f(x)$

For the evaluation of the expansion coefficients of $x^\ell D_q^p f(x)$ as expanded in series of Al-Salam–Carlitz I polynomials, the following theorem is needed.

Theorem 2. *The expansion of the moments of one single Al-Salam–Carlitz I polynomial of any degree in terms of Al-Salam–Carlitz I themselves is given by*

$$x^m U_j^{(\alpha)}(x;q) = \sum_{n=0}^{2m} a_{mn}(j) U_{j+m-n}^{(\alpha)}(x;q), \qquad m \geqslant 0, \quad j \geqslant 0, \tag{11}$$

where

$$a_{mn}(j) = \sum_{i=\max(j-n,0)}^{j} (-\alpha)^{j-i} q^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q b_{m+j-n}^{(m+i)} {}_1\phi_1 \left[\begin{matrix} q^{-(j-i)} \\ 0 \end{matrix} \middle| q; \frac{q}{\alpha}\right], \tag{12}$$

$$b_n^{(m)} = \begin{bmatrix} m \\ n \end{bmatrix}_q \sum_{r=0}^{m-n} \begin{bmatrix} m-n \\ r \end{bmatrix}_q \alpha^r.$$

The following lemma is needed to proceed with the proof of the theorem.

Lemma 1. *It can be shown that the coefficients $a_{mn}(j)$ of (12), satisfy the recurrence relation*

$$\begin{aligned} a_{mn}(j) &= a_{m-1,n}(j) + \beta_{j+m-n} a_{m-1,n-1}(j) + \gamma_{j+m-n+1} a_{m-1,n-2}(j), \\ &\qquad n = 0, \quad 1, \ldots, 2m, \end{aligned} \tag{13}$$

with

$$a_{0,0}(j) = 1, a_{m-1,-\ell}(j) = 0, \quad \forall \ell > 0, \qquad a_{m-1,r}(j) = 0, \quad r = 2m-1, 2m.$$

Proof. Substitution of relation (12) into the rhs of (13), and after performing some slight manipulation, yields the lhs of (13), which completes the proof of lemma 1. □

Proof of Theorem 2. To prove this theorem we proceed by induction. In view of recurrence relation (1), we may write

$$x U_j^{(\alpha)}(x;q) = a_{10}(j) U_{j+1}^{(\alpha)}(x;q) + a_{11}(j) U_j^{(\alpha)}(x;q) + a_{12}(j) U_{j-1}^{(\alpha)}(x;q), \tag{14}$$

and this in turn shows that (11) is true for $m = 1$. Proceeding by induction, assuming that (11) is valid for m, we want to prove that

$$x^{m+1} U_j^{(\alpha)}(x;q) = \sum_{n=0}^{2m+2} a_{m+1,n}(j) U_{j+m-n+1}^{(\alpha)}(x;q). \tag{15}$$

From (14) and assuming the validity for m, we have

$$x^{m+1}U_j^{(\alpha)}(x;q) = \sum_{n=0}^{2m} a_{mn}(j)\big[a_{10}(j+m-n)U_{j+m-n+1}^{(\alpha)}(x;q) + a_{11}(j+m-n)U_{j+m-n}^{(\alpha)}(x;q) + a_{12}(j+m-n)U_{j+m-n-1}^{(\alpha)}(x;q)\big].$$

Collecting similar terms, we get

$$\begin{aligned} x^{m+1}U_j^{(\alpha)}(x;q) &= a_{m0}(j)a_{10}(j+m)U_{j+m+1}^{(\alpha)}(x;q) \\ &\quad + [a_{m1}(j)a_{10}(j+m-1) + a_{m0}(j)a_{11}(j+m)]U_{j+m}^{(\alpha)}(x;q) \\ &\quad + \sum_{n=2}^{2m}[a_{mn}(j)a_{10}(j+m-n) + a_{m,n-1}(j)a_{11}(j+m-n+1) \\ &\quad + a_{m,n-2}(j)a_{12}(j+m-n+2)]U_{j+m-n+1}^{(\alpha)}(x;q) \\ &\quad + [a_{m,2m}(j)a_{11}(j-m) + a_{m,2m-1}(j)a_{12}(j-m+1)]U_{j-m}^{(\alpha)}(x;q) \\ &\quad + a_{m,2m}(j)a_{12}(j-m)U_{j-m-1}^{(\alpha)}(x;q). \end{aligned} \tag{16}$$

Application of lemma 1 given in (13) to equation (16) yields equation (15) and the proof of the theorem is complete.

Note. It is to be noted here that relation (1) is equivalent to

$$x\mathbf{U} = C\mathbf{U},$$

where $\mathbf{U}$ is the vector $\big(U_0^{(\alpha)}, U_1^{(\alpha)}, \ldots\big)^T$ and $C = \big(c_{ij}^{(1)}\big)$ is a tridiagonal matrix whose entries have the form

$$c_{ij}^{(1)} = a_{1,i-j+1}(i), \qquad i, j \geqslant 0,$$

and accordingly relation (11) is equivalent to

$$x^m\mathbf{U} = C^m\mathbf{U},$$

where $C^m = \big(c_{ij}^{(m)}\big)$ is a band matrix of width m whose entries are given explicitly by

$$c_{ij}^{(m)} = a_{m,i-j+m}(i), \qquad i, j \geqslant 0.$$

According to theorem 2, we can state the following theorem which relates the Al-Salam–Carlitz I coefficients of the moments of a general-order q-derivative of an infinitely q-differentiable function in terms of its Al-Salam–Carlitz I coefficients.

Theorem 3. *Assume that $f(x)$, $D_q^p f(x)$ and $x^\ell U_j^{(\alpha)}(x;q)$ have the Al-Salam–Carlitz I expansions (5), (6) and (11) respectively, and assume also that*

$$x^\ell\left(\sum_{i=0}^{\infty} a_i^{(p)}U_i^{(\alpha)}(x;q)\right) = \sum_{i=0}^{\infty} b_i^{p,\ell}U_i^{(\alpha)}(x;q), \tag{17}$$

then the connection coefficients $b_i^{p,\ell}$ are given by

$$b_i^{p,\ell} = \begin{cases} \sum_{k=0}^{\ell-1} a_{\ell,k+\ell-i}(k)a_k^{(p)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & 0 \leqslant i \leqslant \ell, \\ \sum_{k=i-\ell}^{\ell-1} a_{\ell,k+\ell-i}(k)a_k^{(p)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & \ell+1 \leqslant i \leqslant 2\ell-1, \\ \sum_{k=i-2\ell}^{i} a_{\ell,k+2\ell-i}(k+\ell)a_{k+\ell}^{(p)}, & i \geqslant 2\ell. \end{cases} \tag{18}$$

5. Recurrence relations for connection coefficients between Al-Salam–Carlitz I and monic q-polynomials belonging to the q-Hahn tableau

Let $f(x)$ has the expansion (5), and assume that it satisfies the nonhomogeneous linear q-difference equation of order n

$$\sum_{i=0}^{n} p_i(x) D_q^i f(x) = g(x), \tag{19}$$

where $p_0, p_1, \ldots, p_n \neq 0$ are polynomials in x, and the coefficients of Al-Salam–Carlitz I series of the function $g(x)$ are known, then formulae (7), (11) and (18) enable one to construct in view of (19) the linear recurrence relation of order r,

$$\sum_{j=0}^{r} \alpha_j(k) a_{k+j} = \beta(k), \qquad k \geqslant 0, \tag{20}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_r (\alpha_0 \neq 0, \alpha_r \neq 0)$ are polynomials of the variable k.

In this section, we consider the problem of determining the connection coefficients between different polynomial systems. An interesting question is how to transform the Fourier coefficients of a given polynomial corresponding to an assigned orthogonal basis into the coefficients of another basis orthogonal with respect to a different weight function. The aim is to determine the so-called connection coefficients of the expansion of any element of the first basis in terms of the elements of the second basis. Suppose V is a vector space of all polynomials over the real or complex numbers and V_m is the subspace of polynomials of degree less or equal to m. Suppose $p_0(x), p_1(x), p_2(x), \ldots$ is a sequence of polynomials such that $p_n(x)$ is of exact degree n; let $q_0(x), q_1(x), q_2(x), \ldots$ be another such sequence. Clearly, these sequences form a basis for V. It is also evident that $p_0(x), p_1(x), \ldots, p_m(x)$ and $q_0(x), q_1(x), \ldots, q_m(x)$ give two bases for V_m. While working with finite-dimensional vector spaces, it is often necessary to find the matrix that transforms a basis of a given space to another basis. This means that one is interested in the connection coefficients $a_i(n)$ that satisfy

$$Q_n(x) = \sum_{i=0}^{n} a_i(n) P_i(x). \tag{21}$$

The choice of $P_n(x)$ and $Q_n(x)$ depends on the situation. For example, suppose

$$P_n(x) = x^n, \qquad Q_n(x) = (x; q)_n,$$

then the connection coefficients $a_i(n)$ are given by (see Gasper and Rahman (1990))

$$a_i(n) = \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i q^{i(i-1)/2}.$$

If the roles of these $P_n(x)$ and $Q_n(x)$ are interchanged, then we get (see Area *et al* (1999), p 774, equation (3.3))

$$a_i(n) = \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i q^{i(i+1-2n)/2}.$$

It is known that all the polynomials belonging to the q-Hahn tableau (see Koornwinder (1994)) are eigenfunctions of a second-order q-difference equation which can be written as

$$\sigma(x) D_q D_{1/q} y(x) + \tau(x) D_q y(x) + \lambda_{n,q} y(x) = 0, \tag{22}$$

where the expressions of $\sigma(x)$ and $\tau(x)$ are polynomials in x of degree at most 2 and exactly 1, respectively, and $\lambda_{n,q} = [n]_q\left(\frac{1}{2}[1-n]_q\sigma'' - \tau'\right)$ (see Lewanowicz *et al* (2000), formula (3.2)). In equation (22), replacing x by qx gives

$$\sigma(qx)(D_q D_{1/q} y)(qx) + \tau(qx)(D_q y)(qx) + \lambda_{n,q} y(qx) = 0. \tag{23}$$

In view of (3), we can deduce the following three relations:

$$y(qx) = y(x) + (q-1)x D_q y(x), \tag{24}$$

$$(D_q D_{1/q} y)(qx) = q^{-1} D_q^2 y(x), \tag{25}$$

$$(D_q y)(qx) = D_q y(x) + (q-1)x D_q^2 y(x). \tag{26}$$

Substitution of (24)–(26) into (23) gives

$$\tilde{\sigma}(x) D_q^2 y(x) + \tilde{\tau}(x) D_q y(x) + \lambda_{n,q} y(x) = 0, \tag{27}$$

where $\tilde{\sigma}(x) = q^{-1}\sigma(qx) + (q-1)x\tau(qx)$ and $\tilde{\tau}(x) = \tau(qx) + \lambda_{n,q}(q-1)x$.

5.1. *Big q-Jacobi–Al-Salam–Carlitz I connection problem*

The link between monic big q-Jacobi polynomials, $P_n(x; a, b, c; q)$ and $U_i^{(\alpha)}(x; q)$ given by (21) can easily be replaced by a linear relation involving only $U_i^{(\alpha)}(x; q)$ using the big q-Jacobi polynomials q-difference equation, namely,

$$\begin{aligned}&[aq^2(q-1)(qx-1)(bqx-c)D_q^2 - q^2\{(q^{-n} + abq(-1-q+q^n))x \\ &\quad + (-c + a(-1+(b+c)q))\}D_q - q^2[n]_q(abq - q^{-n})]P_n(x; a, b, c; q) = 0.\end{aligned} \tag{28}$$

By substituting

$$P_n(x; a, b, c; q) = \sum_{i=0}^{n} a_i(n) U_i^{(\alpha)}(x; q), \tag{29}$$

and by virtue of formula (17), equation (28) takes the form

$$\begin{aligned}&ac(q-1)^2 q^2 b_i^{2,0} - aq^3(q-1)^2(b+c)b_i^{2,1} + ab(q-1)^2 q^4 b_i^{2,2} \\ &\quad + q^2(q-1)(c - a((b+c)q-1))b_i^{1,0} \\ &\quad + (1-q)q^2(q^{-n} + abq(q^n - q - 1))b_i^{1,1} \\ &\quad + q^{2-n}(q^n-1)(1-abq^{n+1})b_i^{0,0} = 0.\end{aligned}$$

By making use of formulae (11) and (18), we obtain

$$\begin{aligned}&q^{2-n}(q-1)^{-2}(q^n-1)(1-abq^{n+1})a_i(n) - q^{2-n}(q-1)^{-1}[1 + abq^{n+1}(q^n - q - 1)]a_{i-1}^{(1)}(n) \\ &\quad + q^{2-n}(q-1)^{-1}[q^n(a + c - a(b+c)q) \\ &\quad - q^i(1 + abq^{n+1}(q^n - q - 1))(\alpha+1)]a_i^{(1)}(n) \\ &\quad + \alpha(q-1)^{-1}q^{2-n+i}(1-q^{i+1})[1 + abq^{n+1}(q^n - q - 1)]a_{i+1}^{(1)}(n) \\ &\quad + abq^4 a_{i-2}^{(2)}(n) - aq^3[c - b(q^i(1+q)(1+\alpha) - 1)]a_{i-1}^{(2)}(n) \\ &\quad + aq^2[c - q^{i+1}(b + c + (c + b(2+q))\alpha) \\ &\quad + bq^{2i+1}(\alpha + q(1 + \alpha(2 + q + \alpha)))]a_i^{(2)}(n) \\ &\quad + a\alpha q^{i+3}(1-q^{i+1})[c - b(q^{i+1}(1+q)(1+\alpha) - 1)]a_{i+1}^{(2)}(n) \\ &\quad + ab\alpha^2 q^{2i+5}(1-q^{i+1})(1-q^{i+2})a_{i+2}^{(2)}(n) = 0, \qquad i \geqslant 0.\end{aligned} \tag{30}$$

Using formula (7) with (30)—and after some slight manipulation—we obtain the following recurrence relation,

$$\delta_{i0}a_i(n)+\delta_{i1}a_{i+1}(n)+\delta_{i2}a_{i+2}(n)+\delta_{i3}a_{i+3}(n)+\delta_{i4}a_{i+4}(n)=0,\qquad i=n-1,n-2,\ldots,0, \tag{31}$$

where

$$\begin{aligned}
\delta_{i0}&=(q^n-q^i)(abq^{n+i+1}-1),\\
\delta_{i1}&=q^n(1-q^{i+1})[a+c+abq^{2i+1}(1+q)(1+\alpha)\\
&\quad -q^{i-n}(1+\alpha+abq^{2n+1}(1+\alpha)+a(b+c)q^{n+1})],\\
\delta_{i2}&=-(q^{i+1};q)_2[-q^i\alpha-\alpha abq^{2n+i+1}\\
&\quad +q^n(c+aq^{i+1}(q^i(\alpha+q(1+\alpha(2+q+\alpha)))b-(1+\alpha)(b+c)))],\\
\delta_{i3}&=\alpha aq^{n+i+1}(q^{i+1};q)_3[-(b+c)+bq^{i+1}(1+q)(1+\alpha)b],\\
\delta_{i4}&=-ab\alpha^2q^{n+2i+3}(q^{i+1};q)_4,
\end{aligned}$$

with $a_{n+s}(n)=0$, $s=1,2,3$ and $a_n(n)=1$. The solution of (31) is

$$\begin{aligned}
a_i(n)&=\frac{(-1)^i(aq,cq;q)_n}{(abq^{n+1};q)_n}\frac{(q^{-n},abq^{n+1};q)_iq^{\binom{i+1}{2}}}{(aq,cq;q)_i(q;q)_i}\\
&\quad\times\sum_{j=0}^{n-i}\frac{(-q^{i+1})^jq^{\binom{j}{2}}(q^{-n+i},abq^{n+i+1};q)_jr_j(\alpha)}{(q;q)_j(aq^{i+1},cq^{i+1};q)_j}\\
&\quad\times{}_3\phi_2\left[\begin{matrix}q^{-n+i+j},\ abq^{n+i+j+1},\ 0\\ aq^{i+j+1},\ cq^{i+j+1}\end{matrix}\,\middle|\,q;q\right],\qquad i=0,1,\ldots,n,
\end{aligned}\tag{32}$$

where

$$r_j(\alpha)=\sum_{m=0}^{j}\begin{bmatrix}j\\ m\end{bmatrix}_q\alpha^m. \tag{33}$$

The monic big q-Laguerre polynomials, $P_n(x;a,b;q)$, and the monic big q-Jacobi polynomials are related by

$$P_n(x;a,b;q)=P_n(x;a,0,b;q), \tag{34}$$

while the monic q-Hahn polynomials, $Q_n(x;a,b,N;q)$, can be obtained from the monic big q-Jacobi polynomials by using the relation

$$Q_n(x;a,b,N;q)=P_n(x;a,b,q^{-N-1};q), \tag{35}$$

and accordingly, the connection problems monic big q-Laguerre–Al-Salam–Carlitz I and monic q-Hahn–Al-Salam–Carlitz I can be deduced with the aid of relations (34) and (35), respectively, with relation (29).

Corollary 1. *In the connection problem*

$$P_n(x;a,b;q)=\sum_{i=0}^{n}a_i(n)U_i^{(\alpha)}(x;q), \tag{36}$$

the expansion coefficients $a_i(n)$ are given by

$$\begin{aligned}
a_i(n)&=\frac{(-1)^i(aq,bq;q)_n(q^{-n};q)_iq^{\binom{i+1}{2}}}{(aq,bq;q)_i(q;q)_i}\\
&\quad\times\sum_{j=0}^{n-i}\frac{(-q^{i+1})^jq^{\binom{j}{2}}(q^{-n+i};q)_jr_j(\alpha)}{(q;q)_j(aq^{i+1},bq^{i+1};q)_j}{}_3\phi_2\left[\begin{matrix}q^{-n+i+j},\ 0,\ 0\\ aq^{i+j+1},\ bq^{i+j+1}\end{matrix}\,\middle|\,q;q\right],\\
&\quad i=0,1,\ldots,n,
\end{aligned}\tag{37}$$

while in the connection problem

$$Q_n(x; a, b, N; q) = \sum_{i=0}^{n} a_i(n) U_i^{(\alpha)}(x; q), \tag{38}$$

the expansion coefficients $a_i(n)$ are given by

$$\begin{aligned} a_i(n) = {} & \frac{(aq, q^{-N}; q)_n}{(abq^{n+1}; q)_n} \frac{(-1)^i (q^{-n}, abq^{n+1}; q)_i q^{\binom{i+1}{2}}}{(aq, q^{-N}; q)_i (q; q)_i} \\ & \times \sum_{j=0}^{n-i} \frac{(-q^{i+1})^j q^{\binom{j}{2}} (q^{-n+i}, abq^{n+i+1}; q)_j r_j(\alpha)}{(q; q)_j (aq^{i+1}, q^{i-N}; q)_j} \\ & \times {}_3\phi_2 \left[\begin{matrix} q^{-n+i+j}, \ abq^{n+i+j+1}, \ 0 \\ aq^{i+j+1}, \ q^{i+j-N} \end{matrix} \middle| q; q \right], \qquad i = 0, 1, \ldots, n. \end{aligned} \tag{39}$$

The monic q-Meixner polynomials, $M_n(x; b, c; q)$, can be obtained from the monic q-Hahn polynomials by using the relation

$$\lim_{N\to\infty} Q_n(x; b, -b^{-1}c^{-1}q^{-N-1}, N; q) = M_n(x; b, c; q), \tag{40}$$

and in view of the limiting relation

$$\lim_{N\to\infty} \frac{(\alpha q^{-N}; q)_n}{(\beta q^{-N}; q)_n} = (\alpha/\beta)^n, \tag{41}$$

we obtain the following corollary as a consequence of the two relations (38) and (40).

Corollary 2. *In the connection problem*

$$M_n(x; b, c; q) = \sum_{i=0}^{n} a_i(n) U_i^{(\alpha)}(x; q), \tag{42}$$

the coefficients $a_i(n)$ are given by

$$\begin{aligned} a_i(n) = {} & \frac{(-1)^n (bq; q)_n (q^{-n}; q)_i c^{n-i} q^{\binom{i+1}{2}}}{(bq; q)_i (q; q)_i q^{n(n-i)}} \sum_{j=0}^{n-i} \frac{(q^{n+i+1}/c)^j (q^{-n+i}; q)_j q^{\binom{j}{2}} r_j(\alpha)}{(q; q)_j (bq^{i+1}; q)_j} \\ & \times {}_2\phi_1 \left[\begin{matrix} q^{-n+i+j}, \ 0 \\ bq^{i+j+1} \end{matrix} \middle| q; -\frac{q^{n+1}}{c} \right], \qquad i = 0, 1, \ldots, n. \end{aligned} \tag{43}$$

The monic q-Meixner polynomials and the monic q-Charlier polynomials, $C_n(x; a; q)$, are related by

$$M_n(x; 0, a; q) = C_n(x; a; q), \tag{44}$$

and in view of the two identities,

$${}_2\phi_1 \left[\begin{matrix} c, 0 \\ d \end{matrix} \middle| q; z \right] \overset{d=0}{=} {}_1\phi_0 \left[\begin{matrix} c \\ - \end{matrix} \middle| q; z \right]$$

and

$${}_1\phi_0 \left[\begin{matrix} q^{-n} \\ - \end{matrix} \middle| q; z \right] = (zq^{-n}; q)_n, \qquad n = 0, 1, 2, \ldots,$$

(see Koekoek and Swarttouw (1998), p 14, equation (0.5.3)), we obtain the following corollary as a consequence of the two relations (42) and (44).

Corollary 3. *In the connection problem*

$$C_n(x;a;q) = \sum_{i=0}^{n} a_i(n) U_i^{(\alpha)}(x;q), \tag{45}$$

the coefficients $a_i(n)$ are given by

$$a_i(n) = \frac{(-1)^n (q^{-n};q)_i a^{n-i} q^{\binom{i+1}{2}}}{(q;q)_i q^{n(n-i)}} \times \sum_{j=0}^{n-i} \frac{(q^{n+i+1}/a)^j}{(q;q)_j} \frac{(q^{-n+i};q)_j(-q^{i+j+1}a^{-1};q)_{n-i-j}}{q^{-\binom{j}{2}}} r_j(\alpha), \qquad i = 0,1,\ldots,n. \tag{46}$$

The monic q-Krawtchouk polynomials, $K_n(x;p,N;q)$, may be obtained from the monic q-Hahn polynomials by using the limiting relation

$$\lim_{a\to 0} Q_n(x;a,-pa^{-1}q^{-1},N;q) = K_n(x;p,N;q), \tag{47}$$

and in view of the identity

$${}_3\phi_2\left[\begin{matrix} b,c,0 \\ d,e \end{matrix}\,\middle|\, q;z\right] \overset{e=0}{=} {}_2\phi_1\left[\begin{matrix} b,c \\ d \end{matrix}\,\middle|\, q;z\right],$$

and the q-analogues of the Vandermonde summation formula

$${}_2\phi_1\left[\begin{matrix} q^{-n},b \\ c \end{matrix}\,\middle|\, q;q\right] = \frac{(b^{-1}c;q)_n}{(c;q)_n} b^n, \qquad n = 0,1,2,\ldots,$$

(see Koekoek and Swarttouw (1998), p 15, equation (0.5.9)), we get the following corollary as a consequence of the two relations (38) and (47).

Corollary 4. *In the connection problem*

$$K_n(x;p,N;q) = \sum_{i=0}^{n} a_i(n) U_i^{(\alpha)}(x;q), \tag{48}$$

the coefficients $a_i(n)$ are given by

$$a_i(n) = \frac{(-1)^n (q^{-N};q)_n (q^{-n},-pq^n;q)_i (-p^{-1}q^{-n-N};q)_{n-i} q^{\binom{i+1}{2}} (pq^{n+i})^{n-i}}{(-pq^n;q)_n (q^{-N};q)_i (q^{i-N};q)_{n-i} (q;q)_i} \times \sum_{j=0}^{n-i} \frac{(q^{N+1})^j (q^{-n+i},-pq^{n+i};q)_j}{(q;q)_j (-pq^{N+i+1};q)_j} r_j(\alpha), \qquad i = 0,1,\ldots,n. \tag{49}$$

5.2. Little q-Jacobi–Al-Salam–Carlitz I connection problem

In this problem

$$p_n(x;a,b|q) = \sum_{i=0}^{n} a_i(n) U_i^{(\alpha)}(x;q), \tag{50}$$

where $p_n(x;a,b|q)$ are monic little q-Jacobi polynomials, which satisfy the q-difference equation

$$[a(q-1)qx(bq^2x-1)D_q^2 + \{q(-q^{-n}+abq(1+q-q^n))x + 1 - aq\}D_q - q(abq-q^{-n})[n]_q]p_n(x;a,b|q) = 0, \tag{51}$$

the coefficients $a_i(n)$ satisfy the recurrence relation

$$\delta_{i0}a_i(n) + \delta_{i1}a_{i+1}(n) + \delta_{i2}a_{i+2}(n) + \delta_{i3}a_{i+3}(n) + \delta_{i4}a_{i+4}(n) = 0,$$
$$i = n-1, n-2, \ldots, 0, \tag{52}$$

where

$$\delta_{i0} = (q^n - q^i)(abq^{n+i+1} - 1),$$

$$\delta_{i1} = (1-q^{i+1})[q^{n-1} + abq^{n+2i+1}(1+q)(1+\alpha) - q^i(1+\alpha+aq^n+abq^{2n+1}(1+\alpha))],$$

$$\delta_{i2} = -q^{n+i}(q^{i+1};q)_2[-q^{-n}\alpha - a(1+\alpha) - \alpha abq^{n+1} + abq^{i+1}(\alpha+q(1+\alpha(2+q+\alpha)))],$$

$$\delta_{i3} = \alpha aq^{n+i}(q^{i+1};q)_3[-1+bq^{i+2}(q+1)(\alpha+1)], \qquad \delta_{i4} = -ab\alpha^2q^{n+2i+3}(q^{i+1};q)_4,$$

with $a_{n+s}(n) = 0$, $s = 1, 2, 3$ and $a_n(n) = 1$. The solution of (52) is

$$a_i(n) = (-1)^n q^{\binom{n}{2}+i} \frac{(aq;q)_n(q^{-n}, abq^{n+1};q)_i}{(abq^{n+1};q)_n(aq;q)_i(q;q)_i} \sum_{j=0}^{n-i} \frac{q^j(q^{-n+i}, abq^{n+i+1};q)_j}{(q;q)_j(aq^{i+1};q)_j} r_j(\alpha),$$
$$i = 0, 1, \ldots, n. \tag{53}$$

The monic little q-Laguerre polynomials, $p_n(x;a|q)$, and the monic little q-Jacobi polynomials are related by

$$p_n(x;a|q) = p_n(x;a,0|q), \tag{54}$$

while the monic alternative q-Charlier, $K_n(x;b;q)$, can be deduced from the monic little q-Jacobi polynomials by using the limiting relation

$$\lim_{a\to 0} p_n(x;a,-b/aq|q) = K_n(x;b;q), \tag{55}$$

and accordingly, the connection problems monic little q-Laguerre–Al-Salam–Carlitz I and monic alternative q-Charlier–Al-Salam–Carlitz I can be deduced with the aid of relations (54) and (55), respectively, with relation (50).

Corollary 5. *The link between monic little q-Laguerre–Al-Salam–Carlitz I connection problem is given by*

$$p_n(x;a|q) = \sum_{i=0}^{n} a_i(n) U_i^{(\alpha)}(x;q), \tag{56}$$

where

$$a_i(n) = (-1)^n q^{\binom{n}{2}+i} \frac{(aq;q)_n(q^{-n};q)_i}{(aq;q)_i(q;q)_i} \sum_{j=0}^{n-i} \frac{q^j(q^{-n+i};q)_j}{(q;q)_j(aq^{i+1};q)_j} r_j(\alpha), \qquad i = 0, 1, \ldots, n, \tag{57}$$

while the link between monic alternative q-Charlier–Al-Salam–Carlitz I connection problem is given by

$$K_n(x;b;q) = \sum_{i=0}^{n} a_i(n) U_i^{(\alpha)}(x;q), \tag{58}$$

where

$$a_i(n) = (-1)^n q^{\binom{n}{2}+i} \frac{(q^{-n}, -bq^n;q)_i}{(-bq^n;q)_n(q;q)_i} \sum_{j=0}^{n-i} \frac{q^j(q^{-n+i}, -bq^{n+i};q)_j}{(q;q)_j} r_j(\alpha),$$
$$i = 0, 1, \ldots, n. \tag{59}$$

Table 1. Polynomials $\sigma(x)$ and $\tau(x)$ in the q-difference equation (22) for the cases of Al-Salam–Carlitz I, II, q-Laguerre and Stieltjes–Wigert.

Family	$\sigma(x)$		$\tau(x)$
Al-Salam–Carlitz I	$U_n^{(\alpha)}(x;q)$	$(x-1)(x-\alpha)$	$\frac{x-\alpha-1}{1-q}$
Al-Salam–Carlitz II	$V_n^{(\alpha)}(x;q)$	α	$\frac{x-\alpha-1}{q-1}$
q-Laguerre	$L_n^{(\alpha)}(x;q)$	x	$\frac{1-q^{\alpha+1}(x+1)}{(1-q)}$
Stieltjes–Wigert	$S_n(x;q)$	x	$\frac{qx-1}{q-1}$

Table 2. Formulae for the connection coefficients in problem (60) for monic Al-Salam–Carlitz I, II, q-Laguerre and Stieltjes–Wigert polynomials, respectively.

$p_n(x;q)$	$a_i(n) \qquad (0 \leqslant i \leqslant n)$
$U_n^{(\beta)}(x;q)$	$\alpha^{n-i}\frac{(q^{i+1};q)_{n-i}(\beta/\alpha;q)_{n-i}}{(q;q)_{n-i}}$
$V_n^{(\beta)}(x;q)$	$(-1)^n\frac{\beta^{n-i}(q^{-n};q)_i q^{in-\binom{n}{2}}}{(q;q)_i}\times$
	$\sum_{j=0}^{n-i}\frac{(q^n/\beta)^j(q^{i-n};q)_j r_j(\alpha)}{(q;q)_j}\,{}_2\phi_0\left[\begin{matrix} q^{-(n-i-j)},0 \\ - \end{matrix}\middle\vert q;\frac{q^{n-i-j}}{\beta}\right]$
$L_n^{(\beta)}(x;q)$	$(-1)^n\frac{q^{\binom{i+1}{2}}}{q^{(n-i)(n+\beta)}}\frac{(q^{\beta+1};q)_n(q^{-n};q)_i}{(q^{\beta+1};q)_i(q;q)_i}\sum_{j=0}^{n-i}\frac{q^{(n+\beta+i+1)j}(q^{-n+i};q)_j q^{\binom{j}{2}}}{(q;q)_j(q^{\beta+i+1};q)_j}r_j(\alpha)$
$S_n(x;q)$	$(-1)^n\frac{q^{\binom{i+1}{2}}}{q^{n(n-i)}}\frac{(q^{-n};q)_i}{(q;q)_i}\sum_{j=0}^{n-i}\frac{q^{(n+i+1)j}(q^{-n+i};q)_j q^{\binom{j}{2}}}{(q;q)_j}r_j(\alpha)$

Remark 1. The expressions of $\sigma(x)$ and $\tau(x)$ in equation (22) and the formulae of connection coefficients, $a_i(n)$, appearing in the connection problem

$$p_n(x;q) = \sum_{i=0}^{n} a_i(n)U_i^{(\alpha)}(x;q), \tag{60}$$

for most of the remaining monic polynomial families, $\{p_n(x;q)\}$, inside the q-Hahn tableau are summarized in tables 1 and 2.

Remark 2. The expansions and connection coefficients in series of discrete q-Hermite polynomials of the first kind, $h_n(x;q)$, can be obtained directly from those of the Al-Salam–Carlitz I polynomials $U_n^{(\alpha)}(x;q)$, by taking $\alpha = -1$, and taking into account the Gauss identities

$$r_j(-1) = \begin{cases} (q;q^2)_{j/2}, & j \text{ even}, \\ 0, & j \text{ odd}, \end{cases}$$

(see Kac and Cheung (2002), formulae (7.14) and (7.15)).

Remark 3. It is to be noted that one of our goals is to emphasize the systematic character and simplicity of our algorithm, which allows one to implement it in any computer algebra (here the Mathematica (1999) symbolic language has been used).

To end this paper, we wish to report that this work deals with formulae associated with the Al-Salam–Carlitz I coefficients for the moments of $D_q^p f(x)$, $p = 0, 1, 2, \ldots,$ and with the connection coefficients between each family belonging to the q-Hahn tableau and Al-Salam–Carlitz I polynomials. These formulae can be used to facilitate greatly the setting up of the algebraic systems to be obtained by applying the spectral methods for solving q-difference equations with polynomials coefficients of any order.

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